Gravito-Acoustic Waves and Instabilities

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1 Theoretical background

We will first study the behavior of waves in an incompressible gravitating plasma slab. The displacement of the plasma ξ along the direction of the gravitational field is governed by:

$$\frac{d}{dx}\left[\rho_0\omega^2\frac{d\xi}{dx}\right]-k^2\left[\rho_0\omega^2+\rho_0^{'}g\right]\xi=0 \tag{1}$$

with ρ_0 the plasma density profile and ω the eigenfrequencies.

Equation (1) can be rewritten into:

$$\frac{d^2\xi}{dx^2}+\frac{\rho_0^{'}}{\rho_0}\frac{d\epsilon}{dx}-k^2\left[1+\frac{\rho_0^{'}}{\rho_0}\frac{g}{\omega^2}\right]\xi=0 \tag{2}$$

There exists an analytical solution in the case of an isothermal stratified medium ($\rho_0 = e^{-\alpha x}$) with $\alpha = \frac{1}{H}$ where $H = \frac{kT}{mg}$ is the scale height. Substituting this expression into (2), gives:

$$\frac{d^2\xi}{dx^2}-\alpha\frac{d\xi}{dx}+k^2\left[\frac{\alpha g}{\omega^2}-1\right]\xi=0 \tag{3}$$

Introducing the Brunt-Väisälä frequency $N = \alpha g = g^2$ which is some kind of cut-off frequency. The solutions of this linear differential equation are of the form:

$$\xi = Ce^{(\frac{1}{2}\alpha\pm iq)x} \tag{4}$$

with the parameter q:

$$q=\sqrt{k^2\left[\frac{N^2}{\omega^2}-1\right]-\frac{\alpha^2}{4}} \tag{5}$$

With boundary conditions $\xi(0) = \xi(1) = 0$, the parameter q has to satisfy $q = n\pi$ with $n = 1, 2, ...$ This condition gives a dispersion relation for the eigenfrequencies ω.

$$\omega^2=\frac{k^2N^2}{(n\pi)^2+k^2+\frac{\alpha^2}{4}} \tag{6}$$

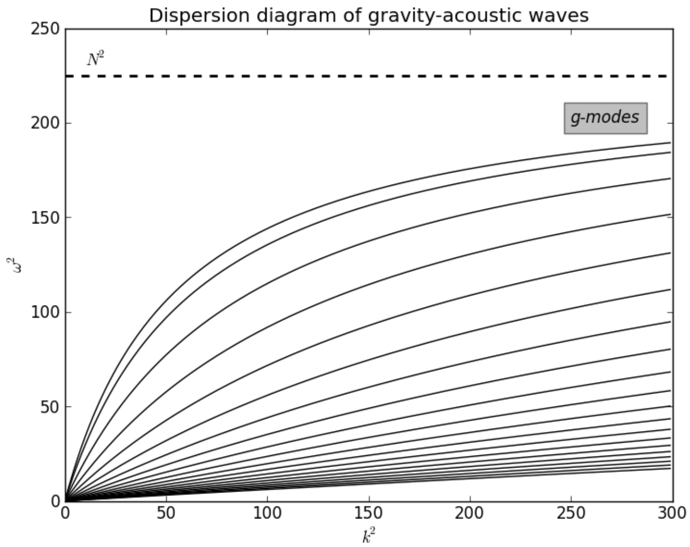

Figure 1: The dispersion relationship for the gravito-acoustic waves with $\alpha = 15$.

This dispersion relationship is plotted in Figure (1) where the Brunt-Väisälä frequency is the maximum attainable frequency.

For these boundary conditions the solutions (4) for the displacements ξ become:

$$\xi = \frac{1}{n\pi} e^{\frac{1}{2}\alpha x} sin(n\pi x) \tag{7}$$

The first three modes are plotted in Figure (2).

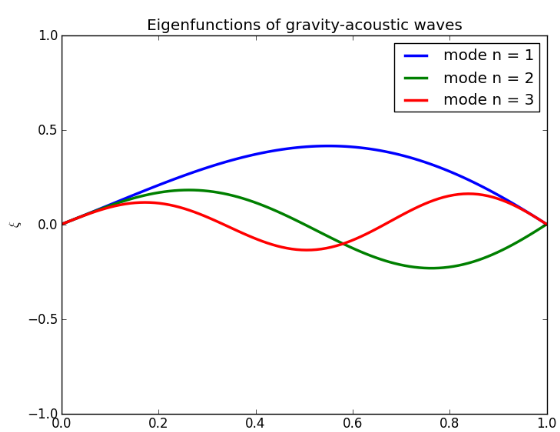

Figure 2: The eigenfunctions for the gravito-acoustic waves with $\alpha = 1$.

2 Numerical implementation

Equation (1) has the form:

$$\frac{d}{dx}\left[P(x,\omega^2)\frac{d\xi}{dx}\right] - Q(x,\omega^2)\xi = 0 \tag{8}$$

which can be split into a set of two first order differential equations:

$$\begin{aligned} \frac{d\xi}{dx} &= \frac{\Phi}{P(x,\omega^2)} \\ \frac{d\Phi}{dx} &= Q(x,\omega^2)\xi \end{aligned} \tag{9}$$

These equations (9) can be integrated numerically with a fourth-order Runge-Kutta method with initial conditions $\xi(0) = 0$ and $\Phi(0) = P(0,\omega^2)$. The general idea for such an integration scheme:

$$\begin{aligned} x_{n+1} &= x_n + h \\ y_{n+1} &= y_n + \frac{h}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right] \end{aligned} \tag{10}$$

$$\begin{aligned} k_1 &= f(x_n, y_n, \omega) \\ k_2 &= f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1, \omega) \\ k_3 &= f(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_2, \omega) \\ k_4 &= f(x_n + h, y_n + hk_3, \omega) \end{aligned} \tag{11}$$

with distance $x \in [0,1]$, the solution vector $y = [\xi, \Phi]$, the function vector $f = [\frac{\Phi}{P}, Q\xi]$ and marching step h.

To ensure that the boundary condition $\xi(1) = 0$ is satisfied, a shooting method is applied which in fact searches for the eigenfrequencies of the system. This is made possible by constructing a residual function which depends on the frequency ω and which returns the displacement of the medium at $x = 1$. In fact we are looking for the zeros of the residual function. A root searching routine brackets the root in an interval where the residual function changes sign. As soon as the bracketing is done, the roots are retrieved through Ridder's method.

The method works as follows. Let say the root is bracketed between $[\omega_1, \omega_2]$ then it is possible to determine the midpoint $\omega_3 = \frac{\omega_1+\omega_2}{2}$ and create a function g of the form:

$$g(\omega) = r(\omega)e^{(\omega-\omega_1)Q} \tag{12}$$

Demanding that the three points $(\omega_1, g(\omega_1))$, $(\omega_2, g(\omega_2))$ and $(\omega_3, g(\omega_3))$ lie on the same line, gives us a condition for Q. With this particular value for Q, an improved root (13) can be reconstructed through linear interpolation based on the points $(\omega_1, g(\omega_1))$ and $(\omega_3, g(\omega_3))$.

$$\omega_4 = \omega_3 \pm (\omega_3 - \omega_1)\frac{r(\omega_3)}{\sqrt{r(\omega_3)^2 - r(\omega_1)r(\omega_2)}} \tag{13}$$

This routine keeps running until the difference between the new and old calculated root is sufficiently small. This criterion is even sharpened with each new mode that requires calculation since the dispersion relations for higher modes lie closer to each other. First a suitable starting region has to be provided in which the root ω needs to be determined. Judging Figure (1), for all wavenumbers k, all the eigenfrequencies lie between zero and the Brunt-Väisälä frequency. This is thus a natural starting choice. The routine starts at the cut-off frequency and moves downward to retrieve all the modes. In the case of an exponential stratified medium, for larger k the root bracketing will go smoother since the dispersion relationships between all modes are more equidistant. In the exceptional case for small k and small density gradients it is advised to strengthen the initial root finding condition.

The numerical result obtained was the same as the analytical solution displayed in Figure (2). This gives us more confidence that the numerical recipe will work for those situations where no analytical solutions are readily available.

3 Waves and Instabilities

Now consider a medium with a linear density profile $\rho_0(x) = 1 + \sigma x$. Before tackling this numerically, we once again need to find a suitable starting area for determining the eigenfrequencies of this particular system. Therefore multiply (8) with ξ and integrate over $\big[0, 1\big]$.

$$\int_0^1 \left[\, \xi(P\xi^{'})^{'} - Q\xi^2 \,\right] dx = \left[\, P\xi\xi^{'} \,\right]_0^1 - \int_0^1 \left[\, P\xi^{'2} + Q\xi^2 \,\right] dx = 0 \tag{14}$$

The boundary vanishes and since for waves ($\omega^2 > 0$) P is positive everywhere, Q must be negative in some region. For instabilities ($\omega^2 < 0$) P is negative everywhere and Q must be positive in some region.

For an exponential stratified medium we get the following condition for waves: $\omega^2 - \alpha g < 0$. This is exactly the same condition as obtained in the previous section. The eigenfrequencies have to be lower than the Brunt-Väisälä frequency.

For a linear density profile we get for waves ($\sigma < 0$): $\omega^2 < \frac{-g\sigma}{1+\sigma x}$ and for instabilities ($\sigma > 0$): $\omega^2 > \frac{-g\sigma}{1+\sigma x}$. We want to pick the maximum possible cut-off frequencies and thus use $x = 1$ for the inequality condition for waves and $x = 0$ for instabilities.

The first 7 modes are calculated numerically for wavenumber $k = 10$ and marching step $h = 0.01$; for both waves ($\sigma = -0.5$) and instabilities ($\sigma = 0.5$).

Gravity waves are density waves which have gravity as their restoring force. A perturbation causes the fluid parcel to oscillate around its initial position. They are non-propagating waves. One can see in Figure (3) that these oscillations have the biggest amplitudes in these regions where the gravitational force has less influence. Larger negative density gradients thus cause bigger oscillations, which in turn originate closer to the boundary $x = 1$. The oscillations are also larger for higher wavenumbers (or smaller wavelengths).

Unlike waves where the fluid parcel oscillates in time around the initial position, instabilities arise from an unstable stratification, where heavier material lies atop lighter. A fluid parcel, once moved away from its initial position, definitely moves far away. Their amplitudes grow with time. These are known as Rayleigh-Taylor instabilities. As in the case of waves, the initial amplitudes are larger in these regions where the gravitational force is less strong. Larger positive density gradients cause a smaller initial amplitude since more mass lies atop the fluid. Also for a larger positive density gradient, the instabilities will arise closer to the boundary $x = 0$. The initial amplitudes are also larger for smaller wavenumbers (or higher wavelengths).

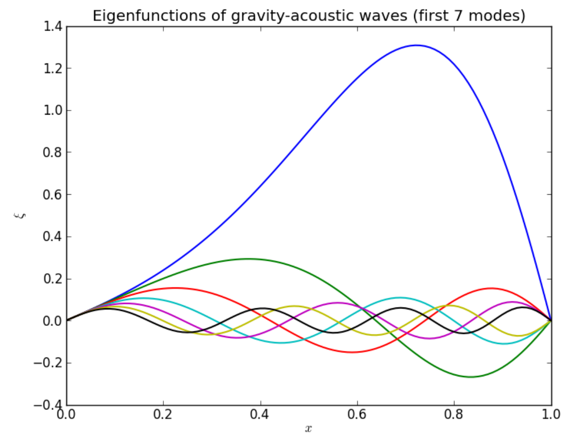

Figure 3: The eigenfunctions for the gravito-acoustic waves with $k = 10$ and $\sigma = -0.5$.

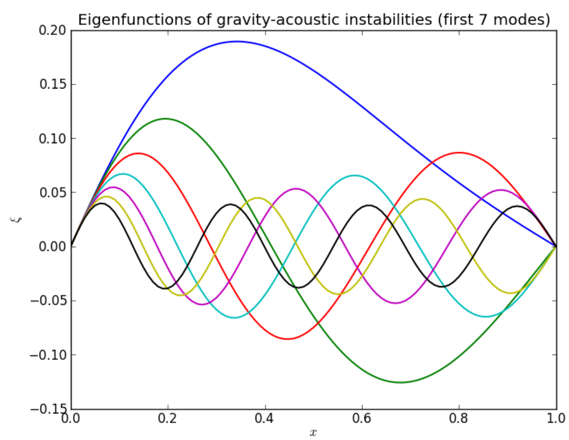

Figure 4: The eigenfunctions for the gravito-acoustic instabilities with $k = 10$ and $\sigma = 0.5$.

4 References

1. Jaan Kiusalaas, *Numerical Methods in Engineering with Python*, 2nd edition, 2010
2. Hans Goedbloed & Stefaan Poedts, *Principles of MHD*, Chapter 7, 2004